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A REMARK ON THE DYNAMIC INVARIANT OR PATH-INDEPENDENT INTEGRAL

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Abstract—The pioneering contributions toward invariant or path-independent integrals of mathematical physics, from the first work by Maxwell in 1873 up to the papers by Brust *et al.* in 1986, are referred to in chronological order, including some brief comments and taking into account the priority question.

It may be appropriate to provide a brief reference to the basic original works on invariant or path-independent integrals in chronological order.

It was Maxwell (1873) who first introduced path-independent integrals into his electromagnetic field theory, while determining the forces of non-electromagnetic nature acting on carriers of electric charges or magnetic vortices. However, for the case of a point charge, the path-independent integrals appeared to be divergent (divergency paradox). To overcome this difficulty an artificial physical procedure based on the concept of interaction energy was invented. This procedure treated in all handbooks on physics is, in no way, connected with Maxwell's path-independent integrals.†

Eshelby (1951) modifed Maxwell's approach as applied to static linear or nonlinear elasticity theory and derived the corresponding path-independent integrals which appeared divergent on cracks, dislocations and many other singularities. Sanders (1960) gave a new mathematical formulation of the Griffith–Irwin criterion for crack extension in linearly elastic solids in the shape of his own path-independent integral and illustrated it on two known examples. From the energy conservation law local approach, Cherepanov (1967, 1968) derived the main invariant integral, while calculating the energy flow into the tip of a moving crack in arbitrary solids with account of any volume and inertia forces. In the particular case of elastostatics, it coincides with the corresponding Eshelby path-independent integral. The work of Cherepanov (1967) firstly advanced as well the solution to crack problems in power-law-hardening solids which was later called the "HRR-approach". (Even the classical Irwin's formula of linear fracture mechanics was strictly derived by the help of invariant integral firstly in the same papers (Cherepanov, 1967, 1968).)

Rice (1968a, b) initiated and provided the primary contribution toward the application of path-independent Eshelby integrals to crack problems in nonlinearly elastic or equivalent elastic–plastic solids. Atkinson and Eshelby (1968), independently of the paper of Cherepanov (1968), calculated the amount of energy flow into the tip of a moving crack in a dynamic elastic case. Günther (1962) and Knowles and Sternberg (1972) were the first who paid attention to the connection between the Noether's theorem and invariant integrals. Utilizing this theorem they derived a new path-independent integral in elastostatics. The general criterion for fracture advanced in the papers by Cherepanov (1967, 1968) was developed later by Landes and Begley (1972), with application to nonlinear elastic or equivalent elastic–plastic fracture, and by Atkinson and Williams (1973), with application to visco-elastic fracture. A comprehensive evaluation of the work before 1972 was given by Williams (1974).

The main crack-tip parameter for the problem of crack propagation is represented, for

† The issue concerned with solving the divergency paradox and which includes the exact definition of Γ-integration and Γ-residue is discussed in detail by Cherepanov in the section entitled "Computation of invariant integrals at singularities" in the Russian volume entitled *Computational Mechanics of Fracture*, MIR Publishers, Moscow. To be published in 1990.

the first time, as the invariant integral in Cherepanov (1967, eqn (1.15)) in the form:

$$\Gamma = \oint_{S_\varepsilon} [(W+T-H)n_1 - t_i u_{i,1})]\,\mathrm{d}s, \quad (i = 1,2). \tag{1}$$

Here, W is the strain work per unit volume, T is the kinetic energy per unit volume, H is the work of external volume forces per unit volume, $t_i = s_{ij}n_j$ are tractions (s_{ij} are stresses), u_i are displacements, n_i are the components of outer unit vector normal to S_ε, and S_ε is a small contour enclosing the crack-tip and moving together with it. The contour S_ε is assumed to be located in "superfine structure" of the crack-tip where the steady state holds, i.e. the stress and strain field is self-similar and independent of time in the moving coordinate system.† This expression is valid for the case of dynamic crack propagation in a dynamically loaded body with arbitrary inelastic properties such as elastic–plastic and/or visco-plastic. The validity is also discussed in Cherepanov (1968), and many applications of this concept are contained in the books by Cherepanov (1979, 1983, 1987).

In the particular case of quasi-static elastic bodies, when $T = 0$, $H = 0$ and $W = U$, where U is the elastic potential per unit volume, the Γ integral (1) is reduced to the J integral introduced by Eshelby (1951) and applied, for the first time, by Rice (1968a, b) to the calculations. Later Landes and Begley (1972) and other western researchers used the J integral as the crack-tip parameter. This parameter appeared fruitful in predicting crack initiation and small amounts of the crack growth in metals.

After the work Cherepanov (1967) completed, it became plausible that the Γ parameter (1) is the unique universal crack-tip parameter for any loading history and for arbitrary materials. To utilize this parameter, however, it is necessary to overcome experimental difficulties, as well as the difficulty in the computation of the Γ parameter for a moving crack-tip using the combined experimental/numerical stress–strain analysis (Cherepanov, 1979). Recently, these barriers were successfully overcome in the papers by Atluri (1982), Nishioka and Atluri (1983, 1984), Atluri *et al.* (1984a, b), Brust *et al.* (1985, 1986) and Brust and Atluri (1986). In these papers, the Γ parameter (1) is redesignated as the T^* integral without citation of the original work in Cherepanov (1967). In spite of this oversight, it is a pleasure to see the outstanding success of this approach. A great advancement in fracture science has taken place since 1967!

The original, simplest idea, that $\Gamma = \Gamma_c$ = a material constant independent of time and loading path, has been experimentally verified and found valid for the prediction of both small and large amounts of crack growth in metals by creep and by complex loading paths of the elastic–plastic behavior. These are brilliant experimental/numerical achievements by Brust, Nishioka, Atluri, McGowan and Nakagaki!

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† As shown in Cherepanov (1967), the parameter (1) holds for any unsteady growth of a crack in arbitrary solids if the crack tip is a certain singularity.

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